REGULARIZATION IN THE REDUCTION OF THE HEAT FLUX TO THE SURFACE OF A BODY WITH NONLINEAR CONDUCTION

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The surface heat flux may be deduced from the time course of the temperature within a body.

Often the only way to measure the heat flux into a material is to measure the temperature within the material.

Small errors in the measured temperatures may represent very large errors in the calculated heat flux; for instance, the body largely fails to respond to high-frequency fluctuations in the heat flux, so minor errors δ_{meas} in the observed temperature may be considered as consequences of fast alternating flux components of arbitrarily large amplitude. Therefore, the problem is to be considered as incorrectly formulated unless additional information is available; a stable solution may not be attained if the input data are perturbed.

An efficient method of constructing a stable approximation for such cases is based on qualitative a priori information [3-5, 8] (regularization). In particular, surface-temperature determination has been considered [1] for the case where the thermal characteristics of the body are independent of temperature.

We consider the propagation of heat in a rod (0, R) with an adiabatic side surface and adiabatic right-hand end where the physical parameters are dependent on temperature:

$$\frac{\partial}{\partial x} \left[\lambda(t) \frac{\partial t}{\partial x} \right] = c(t) \frac{\partial t}{\partial \tau},$$

$$t(x, 0) = t_0(x), \ 0 < \tau < T, \ 0 < x < R,$$

$$\frac{\partial t}{\partial x}\bigg|_{x=R} = 0,$$

$$\lambda(t) \frac{\partial t}{\partial x}\bigg|_{x=0} = q(\tau),$$

(1)

where $q(\tau)$ is the heat flux supplied to the left-hand end, $\lambda(t) \geq \lambda > 0$, $c(t) \geq c > 0$, $t_0(x)$, $\lambda(t)$, and $c(t)$ may be taken as a known smooth function of the argument.

If, for example, $q(\tau)$ is known, then (1) is completely defined, and the temperature can be calculated for any specified point x_1 on the rod ($0 \leq x_1 \leq R$); then we have defined the operator $A[q] \equiv t(x_1, \tau)$, where $t(x_1, \tau)$ is the temperature at point x_1.

We are here interested in the problem inverse to (1), namely, determination of $q(\tau)$ subject to additional specification of $t(\tau)$ at point x_1, which is accessible to measurement. This differs from the treatment of [1] in that the operator $A[q]$ is nonlinear and correspondingly does not have an explicit analytical representation; this can be considered as an extrapolation problem (since we know also the value

$\partial t/\partial x|_{x=R} = 0$), and then it is readily shown that the solution $\bar{q}$ is unique when $\bar{t}_{meas}$ is exactly given.

The attempt to invert (1) directly is equivalent to solving the equation

$$t_{meas} - A[q] = 0. \tag{2}$$

The problem of (2) is nonlinear in this formulation, since the coefficients in the equation are dependent on temperature, while the operator $A[q]$ is given inexplicitly (an algorithm).

The algorithm of [3-6] is reasonably general and applicable to nonlinear inverse problems; the performance of the algorithm is unrelated to the requirement for explicit specification of $A[q]$, or, in other words, one does not need to reduce the problem to an integral equation of the first kind.

It is thus possible to use the above general algorithm, in particular, in the nonlinear theory of thermal conduction, where a difference from [1] is that the Green's function does not allow of analytical representation. Quasiinversion [2] differs from regularization in not requiring, in general, that the approximation converges to the exact solution for $\delta_{meas} \to 0$.

Consider the functional

$$F^{\alpha}[q] \equiv \|A[q] - t_{meas}\|^2 + \alpha \|q'\|^2 \tag{3}$$

and the determination of q^{α} that produces the minimum; it is clear that q^{α} is the solution to (2) for $\alpha = 0$ if t_{meas} is exactly specified, while for $\alpha > 0$ it represents the solution to some smooth problem, which is meaningful for any t_{meas} and is stable with respect to δ_{meas}.

Roughly speaking, for all q that satisfy $\|A[q] - t_{meas}\| < \delta_{meas}$ one should select the smoothest as the solution; then the error arising from the smoothing term $\alpha \|q'\|^2$ should be of the order of the error of measurement, i.e., $\alpha = 0(\delta_{meas}^2)$ [4]. Smoothness in the solution is the qualitative information used here. We minimize $F^{\alpha}[q]$ via an iterative sequence of turning points in quadratic functionals:

$$F_n[q_{n+1}] = \|A[q_n] + A_q'[q_n](q_{n+1} - q_n) - t_{meas}\|^2 + \alpha \|q_{n+1}'\|^2, \tag{4}$$

where $A_q'[q]$ is the operator derivative and q_0 may be specified largely arbitrarily.

The following choice of α is used: one specifies the sequence $\{\alpha_p, \alpha_{p+1} = \mu \alpha_p, \mu < 1\}$, with α_0 fairly large, and from $\{q^{\alpha_p}\}$ we select the solution q^{α} that realizes $\min \|q^{\alpha_p} - q^{\alpha_{p-1}}\|$; then $\alpha_p = \alpha$ and q^{α} is the required approximate flux. The initial value q_0 for $\alpha = \alpha_{p+1}$ is taken as equal to q^{α_p}, while for $p = 0$ we take $q_0 \equiv 0$, which provides for convergence in the iteration of (4).

In the numerical realization, the t_{meas}, q, q', and $A[q]$ were replaced by the usual grid analogs [7] on the grid $\{x_i = (i + 1/2)h_x, \tau_j = jh_\tau\}$, $0 \le i \le N$, $0 \le j \le M$, $h_x = R/N$, $h_\tau = T/M$, the choice not being of major importance.

The operator derivative $A_q'[q]$ was approximated by a Jacobi matrix, as in [8]; column k is

$$\frac{A[q(j) + \Delta\delta_{jk}] - A[q(j)]}{\Delta},$$

where δ_{jk} is the Kronecker symbol and Δ is the increment in component k of q.

As $A[q]$ has to be determined repeatedly in order to construct the Jacobi matrix, machine-time economy is necessary in the grid analog for $A[q]$.

The algorithm has been tested on a model involving recovery of the heat flux at the end of a rod of length $R = 5$ mm for the interval $0, T = 5$ sec with the following temperature dependence of the factors: $\lambda(t) = 0.75 \cdot 10^{-6}$ kcal/m$\cdot$sec$\cdot$deg$^2 \times t$, $c(t) = 60$ kcal/m$^3 \cdot$deg $- 2.20$ kcal/m$^3 \cdot$deg$^2 \times t$, which means that $\lambda(t)/c(t)$ varies by over a factor 2. Errors in the input data of 2 and 0.02%, respectively, gave errors of 4 and 0.2% in the recovery.

The algorithm may thus be used for calculating actual heat fluxes.

NOTATION

t, temperature; q, heat flux; λ, c, thermal conductivity and specific heat; δ_{meas}, error of temperature measurement; A, direct solution operator; a, regularization parameter.

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